Laplace transform

Syllabus:-

Definition, transform of elementary functions, inverse transforms, transform of derivations, differentiation and integration of transforms, solution of differential equations. Difference between Laplace and Fourier transform.

Laplace transform:-

If $f(t)$ be the real valued function of $'t'$. Then Laplace transform of $f(t)$ denoted by $L\{f(t)\}$ or $F(s)$ is defined by,

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Where s may be real or complex number.

Here $f(t)$ can be given by

$$f(t) = L^{-1}\{F(s)\}$$

Then $f(t)$ is called as inverse Laplace transform of $F(s)$.

Basic properties of Laplace transform

1) Laplace transform of the sum or difference of time function is equal to the sum or difference of Laplace transform of the individual time functions.

$$L\{f_1(t) + f_2(t) + \dots \dots \dots \dots . + f_n(t)\} = F_1(s) + F_2(s) + \dots \dots \dots + F_n(s)$$

2) Laplace transform of product of constant and time function is equal to product of constant and the Laplace transform of the time function.

$$L\{a\, f(t)\} = a\, F(s)$$

3) Linear property : If $F_1(s)$ and $F_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$ respectively. Then

$$L\{a_1\, f_1(t) + a_2\, f_2(t)\} = a_1\, F_1(s) + a_2\, F_2(s)$$

Where a_1 and a_2 are constants.

4) First shifting property : $L\{e^{at}\, f(t)\} = F(s-a)$

5) Change of scaling property : $L\{\, f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Prof. Avadhut Surekha Prakash Manage,
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

Laplace transform of elementary functions

1) $L\{ a\} = \frac{a}{s}$; where a is constant

Sol^n :-

$$L\{a\} = \int_0^\infty e^{-st} a\, dt = -a\left[\frac{e^{-st}}{s}\right]_0^\infty = -\frac{a}{s}\left[e^{-s\, X\, \infty} - e^{-s\, X\, 0}\right]$$

$$= -\frac{a}{s}\,[0 - 1] = \frac{a}{s}$$

2) $L\{ e^{at}\} = \frac{1}{s-a}$; where a is constant

Sol^n :-

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = -\left[\frac{e^{-(s-a)t}}{(s-a)}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}\left[e^{-(s-a) X\, \infty} - e^{-(s-a)\, X\, 0}\right] = -\frac{1}{(s-a)}[0 - 1]$$

$$= \frac{1}{(s-a)}$$

3) $L\{ 1\} = \frac{1}{s}$

Sol^n :-

$$L\{1\} = \int_0^\infty e^{-st} 1\, dt = -\left[\frac{e^{-st}}{s}\right]_0^\infty = -\frac{1}{s}\left[e^{-s\, X\, \infty} - e^{-s\, X\, 0}\right]$$

$$= -\frac{1}{s}\,[0 - 1] = \frac{1}{s}$$

4) $L\{ t\} = \frac{1}{s^2}$

Sol^n :-

$$L\{t\} = \int_0^\infty e^{-st} t\, dt = -\frac{t}{s}\left[e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= -\frac{t}{s}\left[e^{-s\, X\, \infty} - e^{-s\, X\, 0}\right] - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{t}{s}\,[0 - 1] - \frac{1}{s^2}\,[0 - 1]$$

$$= \frac{t}{s} + \frac{1}{s^2}$$

5) $L\{ t^n\} = \frac{n!}{s^{n+1}}$

Sol^n :-

$$L\{ t^n\} = \int_0^\infty e^{-st} t^n\, dt = -\frac{1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1} dt$$

Prof. Avadhut Surekha Prakash Manage,
D. M. S. Mandal’s Bhaurao Kakatkar College, Belgaum

$$= 0 + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1} dt$$

$$= +\frac{n}{s}\left[-\frac{1}{s}[t^{n-1}e^{-st}]_0^\infty + \frac{(n-1)}{s}\int_0^\infty e^{-st}\, t^{n-2} dt\right]$$

$$= \frac{n}{s}\left[0 + \frac{(n-1)}{s}\int_0^\infty e^{-st}\, t^{n-2} dt\right]$$

$$= \frac{n(n-1)}{s}\int_0^\infty e^{-st}\, t^{n-2} dt$$

… … … … … … ….

… … … … … … …

$$= \frac{n(n-1)(n-2)\ldots\ldots 1}{s^n}\int_0^\infty e^{-st}\, dt$$

$$= \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}$$

Other method

$$L\{\, t^n\} = \int_0^\infty e^{-st} t^n\, dt$$

Let $= r$, dt= $\frac{dr}{s}$

$$= \int_0^\infty e^{-r}\left(\frac{r}{s}\right)^n \frac{dr}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-r} r^n\, dr = \frac{n!}{s^{n+1}} \qquad \because \int_0^\infty e^{-r} r^n\, dr = n!$$

6) $L\{\sin at\} = \frac{a}{s^2 + a^2}$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\ dt = \frac{a}{s^2 + a^2}$$

$\int_0^\infty e^{-st} \sin at\ dt$ can be found by substituting $\sin at = \frac{e^{iat} - e^{-iat}}{2i}$

7) $L\{\cos at\} = \frac{s}{s^2 + a^2}$

$$L\{\sin at\} = \int_0^\infty e^{-st} \cos at\ dt = \frac{s}{s^2 + a^2}$$

$\int_0^\infty e^{-st} \sin at\ dt$ can be found by substituting $\cos at = \frac{e^{iat} + e^{-iat}}{2}$

Prof. Avadhut Surekha Prakash Manage,
D. M. S. Mandal’s Bhaurao Kakatkar College, Belgaum

Laplace transform of few other functions

1) $L\{e^{at}\, t^n\} = \frac{n!}{(s-a)^{n+1}}$

2) $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$

3) $L\{e^{at} \cos bt\} = \frac{(s-a)}{(s-a)^2 + b^2}$

4) $L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$

5) $L\{e^{at} \cosh bt\} = \frac{(s-a)}{(s-a)^2 - b^2}$

Inverse Laplace transform

If $L\{f(t)\}$ or $F(s)$ is Laplace transform of $f(t)$, then $L^{-1}\{f(t)\} = F(s)$. Here L^{-1} is called inverse Laplace transform.

Basics Laplace inverse transforms

1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2) $L^{-1}\left\{\frac{1}{(s-a)}\right\} = e^{at}$

3) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

4) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!}$

5) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

6) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

7) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

8) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

9) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$

10) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

11) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$

12) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$

Complicated inverse Laplace transforms can be found by partial fraction method.

Laplace transform of a derivative

Let $f'(t)$ be the first derivative of $f(t)$. Then

$$L\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt = [e^{-st}\, f(t)]_0^\infty + s\int_0^\infty e^{-st}\, f(t)dt$$

$$= [e^{-s\,X\,\infty}\, f(\infty) - e^{-s\,X\,0}\, f(0)] + sF(s)$$

$$= -f(0) + s\,F(s)$$

Prof. Avadhut Surekha Prakash Manage,
D. M. S. Mandal’s Bhaurao Kakatkar College, Belgaum

$$= s\,F(s) - f(0)$$

In general for n^{th} derivative is given by

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \dots \dots \dots . f^n(0)$$

Derivative of Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Differentiating on both side wrt s, we get

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty f(t)\,\frac{d}{ds}(e^{-st})\,dt$$

$$= \int_0^\infty f(t) - t\,e^{-st}\;dt$$

$$F'(s) = L\{-tf(t)\}$$

Differentiating on both side wrt s, we get

$$F''(s) = -\frac{d}{ds}\int_0^\infty f(t)t\,e^{-st}\;dt$$

$$= -\int_0^\infty f(t)t\,\frac{d}{ds}e^{-st}\;dt = \int_0^\infty f(t)t^2\,e^{-st}\;dt = L\{(-t)^2 f(t)\}$$

In general n^{th} derivative of $F(s)$ is

$$F^n(s) = L\{(-t)^n f(t)\}$$

Integration of Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Integrating both side, we get

$$\int_s^\infty F\,(s)\,ds = \int_s^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st} ds\right) f(t)\,dt = -\int_0^\infty \frac{1}{t}\,[e^{-st}\,]_s^\infty f(t)\,dt$$

$$= \int_0^\infty \frac{1}{t} e^{-st} f(t)\,dt = L\left\{\frac{f(t)}{t}\right\}$$

$$\int_s^\infty F\,(s)\,ds = L\left\{\frac{f(t)}{t}\right\}$$

Prof. Avadhut Surekha Prakash Manage,
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

Solution of differential equation

Consider the second order differential equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = r(t)$$

Using Laplace transform,

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \dots\dots\dots f^n(0)$$

We get,

$$s^2 y(s) - s\, y\,(0) - y'(0) + a[s\, y(s) - y\,(0)] + by(s) = R(s)$$

$$(s^2 + as + b\ )y(s) - (s + a)y\,(0) - y'(0) = R(s)$$

$$(s^2 + as + b\ )y(s) = (s + a)y\,(0) + y'(0) + R(s)$$

$$y(s) = \frac{(s + a)y\,(0) + y'(0)}{(s^2 + as + b\ )} + \frac{R\,(s)}{(s^2 + as + b\ )}$$

$$L(y) = \frac{(s + a)y\,(0) + y'(0)}{(s^2 + as + b\ )} + \frac{R\,(s)}{(s^2 + as + b\ )}$$

$$y = L^{-1}\left[\frac{(s + a)y\,(0) + y'(0)}{(s^2 + as + b\ )} + \frac{R\,(s)}{(s^2 + as + b\ )}\right]$$

Relation between Fourier and Laplace transform

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\, L\left\{\frac{f(t)}{e^{-st}}\right\}$$

Difference between Fourier and Laplace transform

Sl. No.	Fourier transform	Laplace transform
1	$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt$	$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$
2	Fourier transform needs Dirichlet conditions.	Laplace transform doesn't need any conditions.
3	Fourier transform does a complex transform on real data.	Laplace transform does a real transform on complex data.
4	Fourier transform converts time varying function in the frequency domain.	Laplace converts time varying function in the integral domain.
5	In Fourier transform, function can have finite number discontinuities in each period	In Laplace transform, function should have definite value over the range.
6	Fourier transform is used in communication system	Laplace transform is used in control system.

Prof. Avadhut Surekha Prakash Manage,
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum